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Extremal, Orthogonality, and Convergence Properties of Multidimensional Splines

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INTRODUCTION

Although one-dimensional[1] splines were introduced by Schoenberg [1] in 1946, their most important intrinsic properties were not realized until more than a decade later. The ***minimum curvature*** property for the cubic spline was discovered by Holladay [2] in 1957; and in 1962 Walsh, Ahlberg, and Nilson [3] established the ***best approximation*** property as well as the simpler convergence properties of cubic splines. An analysis of the more subtle convergence properties of cubic splines by Ahlberg and Nilson [4] followed in 1963. Since then a number of advances [5-9] have extended these properties to general one-dimensional splines of odd degree.

In contrast, progress has been much slower in obtaining similar properties for higher-dimensional splines. The first published results in this direction appeared in a paper by Birkhoff and Garabedian [10] in 1961. The interpolation functions obtained by them, however, lack continuity in some second partial derivatives along mesh lines. DeBoor [11] in 1962 established, for an important set of boundary conditions, the existence and uniqueness of a piecewise bicubic, $S(s, t)$, with $\partial^4 S(s, t)/\partial s^2 \partial t^2$ continuous which, as in the case of Birkhoff and Garabedian, interpolates to prescribed values at the mesh points. In neither case were minimum curvature, best approximation, or convergence properties established. In this paper we generalize one-dimensional spline theory to higher dimensions in a manner which preserves these properties.

ONE-DIMENSIONAL SPLINES

The theory of multidimensional splines depends heavily on the theory of one-dimensional splines. A one-dimensional spline of degree $2k + 1$ on the

[1] *One-dimensional* is used to indicate that the domain of definition is one-dimensional.

closed unit interval I is a function in $C^{2k}(I)$ which is defined by a mesh Δ: $0 = t_0 < t_1 < \cdots < t_{i_N} = 1$, and a set of complex numbers $\{f_i\}$ ($i = 0, 1, \cdots i_N$) to which it interpolates at the mesh points. In each of the intervals $t_{i-1} \leqslant t \leqslant t_i$ ($i = 1, 2, \cdots, i_N$) it coincides with a polynomial of degree $2k + 1$ in t. A one-dimensional spline will be denoted by $S_N(t)$ or, when the set of values $\{f_i\}$ to which it interpolates at the mesh points is emphasized, by $S_N(\{f_i\}_i; t)$. The degree of $S_N(t)$ will be clear from the context in cases where it is important.

It is advantageous to separate one-dimensional splines into classes dependent on the boundary conditions they are required to satisfy at the ends of the unit interval. Three particularly important sets of boundary conditions are

I: at $t = 0$ and $t = 1$, $S_N^{(j)}(t)$ is prescribed for $j = 1, 2, \cdots k$;

II: at $t = 0$ and $t = 1$, $S_N^{(j)}(t)$ is prescribed for $j = k + 1, k + 2, \cdots, 2k$;

III: $S_N(t)$ is periodic on I.[2]

Splines satisfying these conditions will be referred to as Type I splines, Type II splines, and periodic splines, respectively. More generally, any complex-valued function on I satisfying one of these sets of boundary conditions will be referred to as a function of the corresponding type. When all the boundary values imposed on a Type I function or a Type II function are zero, the function will be termed a Type I′ function or a Type II′ function, respectively.

The family of all Type I functions on the interval is partitioned into equivalence classes if one regards two such functions as being equivalent if their difference is a Type I′ function. Similarly one can form equivalence classes of Type II functions. For $i_N \geqslant k$, one-dimensional spline theory has established for each mesh Δ: $0 = t_0 < t_1 < \cdots < t_{i_N} = 1$, and for each set of complex numbers $\{f_i\}$ ($i = 0, 1, \cdots, i_N$) the existence and uniqueness, in each of these equivalence classes, of a spline of degree $2k + 1$ which interpolates to $\{f_i\}$ at the mesh points. Similar results are true for periodic functions, although they are analogous to a single equivalence class.

Minimum curvature properties have been established for periodic functions and for each equivalence class of Type I functions. Best approximation properties have been established for these classes of functions and also for each equivalence class of Type II functions. Generally the minimum curvature or best approximation is only within the equivalence class; there are, however, some important exceptions. We illustrate this by two examples.

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[2] We shall say a function $f(t) \in C^k(I)$ is periodic if it has an extension $\bar{f} \in C^k(-\infty, \infty)$ and $\bar{f}$ has period 1.

EXAMPLE 1. Let $\{f_i\}$ $(i = 0, 1, \cdots, i_N)$ be a set of complex numbers such that $f_0 = f_{i_N}$ and let $\Delta : 0 = t_0 < t_1 < \cdots < t_{i_N} = 1$ be a mesh on I. Then the minimum curvature property of periodic cubic spline asserts that, of all periodic functions $g(t) \in C^2(I)$ which interpolate to $\{f_i\}$ at these mesh points, the periodic cubic spline $S_N(\{f_i\}_i; t)$ minimizes the measure of curvature

$$\int_0^1 | g''(t) |^2 \, dt.$$

If we drop the requirement that the interpolating functions $g(t)$ be periodic, the minimum curvature property for Type II′ functions asserts the existence of a Type II′ spline $\hat{S}_N(\{f_i\}_i; t)$ on Δ such that

$$\int_0^1 | S_N''(t) |^2 \, dt - \int_0^1 | \hat{S}_N''(t) |^2 \, dt = \int_0^1 | S_N''(t) - \hat{S}_N''(t) |^2 \geqslant 0.$$

Thus, unless $S_N(t) \equiv \hat{S}_N(t)$, which is generally not the case, the inequality is strict and our measure of curvature is reduced when $S_N(t)$ is replaced by $\hat{S}_N(t)$.

EXAMPLE 2. Let $g(t) \in C^2(I)$ be given together with a mesh Δ:

$$0 = t_0 < \cdots < t_{i_N} = 1$$

on I. The best approximation property for Type I splines asserts that in each Type I equivalence class the Type I cubic spline of interpolation to $g(t)$ on Δ minimizes the measure of approximation

$$\int_0^1 | g''(t) - S_N''(t) |^2 \, dt$$

over all cubic splines on Δ in the same Type I equivalence class. The Type I spline of interpolation to $g(t)$ on Δ which is in the same equivalence class as $g(t)$ improves the measure of approximation, usually in a strict sense.

Analogous results hold for splines of higher odd degree. The effect of adding more mesh points on measures of curvature and approximation is to cause them, respectively, to increase and decrease monotonically. This is a direct consequence of the orthogonality properties of one-dimensional splines which are examined in [12]. It is worth noting that many splines of mixed type possess minimum curvature and best approximation properties. Since we shall make considerable use in this paper of the convergence properties of one-dimensional splines, we shall review them briefly.

When the minimum curvature property is valid, convergence to an approximated function $F(t) \in C^{k+1}(I)$ and its first k derivatives can be shown for any sequence of splines of degree $2k + 1$ defined by an imbedded sequence of meshes with mesh gauge[3] going to zero and which interpolate to $F(t)$ at mesh points. Mean-square convergence to $F^{(k+1)}(t)$ also can be shown for such a sequence of splines. The requirement that $F(t) \in C^{k+1}(I)$ can be weakened slightly since it is only needed that $F^{(k+1)}(t)$ exist and that for any $t_0 \in I$

$$\int_{t_0}^{t} F^{(k+1)}(t)\, dt = F^{(k)}(t) - F^{(k)}(t_0).$$

If the sequence of meshes has the additional property of being asymptotically uniform,[3] uniform convergence to $F^{(j)}(t)$, $j = k + 1,\ k + 2, \cdots, 2\,k$ on any closed subset of I contained in the interior of I can be shown for most sets of boundary conditions. In the periodic case there is uniform convergence on I itself. In establishing these properties the minimum curvature property is not used.

Partial Splines

Before investigating higher-dimensional splines, we first consider complex-valued partial splines defined on the cartesian product, Y, of a measure space (X, μ) and the closed unit interval I. These partial splines are of considerable importance in themselves. The term, *partial spline*, refers to the fact that for each $p \in X$ a partial spline is a one-dimensional spline defined on I. Thus, a partial spline can be regarded not only as a complex-valued function defined on Y, but also as a one-parameter family of one-dimensional splines defined on I where integration with respect to the parameter is permissible.

We shall restrict ourselves principally to the periodic case; with few exceptions and minor modifications the arguments apply equally well to the non-periodic case, provided appropriate boundary conditions are imposed which preserve the basic identities and boundedness properties required. Aside from occasional comments concerning splines of higher odd degree, we shall restrict ourselves to partial cubic splines. Generalization to splines of higher odd degree proceeds as in [9] and [12]. The details involved in treating complex-valued splines, rather than real-valued splines, are slight and can be found in [12]. Although we shall assume our splines to be complex-valued, we shall restrict our proofs to the real case.

[3] The mesh gauge of a mesh is the maximum distance between adjacent points. A sequence of meshes $\{\Delta_N\}$ ($N = 0, 1, \cdots$) is asymptotically uniform if

$$\lambda_N = \max_j \left| \frac{t_j - t_{j-1}}{t_{j+1} - t_{j-1}} - 1/2 \right| \to 0 \qquad \text{as} \qquad N \to \infty.$$

A partial spline will be denoted by $S_N(p, t)$ and will be associated with a mesh: $0 = t_0 < t_1 < \cdots < t_{i_N} = 1$. It will interpolate at each point $(p, t_i) \in Y$ $(i = 0, 1, \cdots, i_N)$ to a prescribed value $f_i(p)$. Thus, at each t_i, $S_N(p, t)$ coincides with a function $f_i(p)$. When we wish to indicate this specifically, we shall employ the notation $S_N(\{f_i(p)\}_i; p, t)$, now for a partial spline.

For fixed $p \in X$, $S_N(p, t)$ is simply a one-dimensional spline on I to which established theory applies. Consequently, minimum curvature, best approximation, and convergence properties hold for fixed p. In particular, it is true that for each $p \in X$, of all periodic functions $f(p, t) \in C^2(I)$ which interpolate to prescribed values $f_i(p)$ at $t = t_i$ $(i = 0, 1, \cdots, i_N)$, $S_N(\{f_i(p)\}_i; p, t)$ minimizes

$$\int_0^1 \left| \frac{\partial^2}{\partial t^2} f(p, t) \right|^2 dt.$$

As a consequence, $S_N(\{f_i(p)\}_i; p, t)$ minimizes

$$\int_X \int_0^1 \left| \frac{\partial^2 f(p, t)}{\partial t^2} \right|^2 dt d\mu.$$

From the Fubini theorem it follows that $S_N(\{f_i(p)\}_i; p, t)$ minimizes

$$\left\| \frac{\partial^2}{\partial t^2} f(p, t) \right\|_Y$$

where

$$\| g \|_Y^2 = \int_Y | g |^2 \, d\sigma,$$

σ being the product measure induced by μ and dt on Y. This is the *minimum curvature* property for partial cubic splines.

Using parallel arguments we can establish the *best approximation* property for periodic partial cubic splines which asserts: given a periodic function $f(p, t)$ (periodic in t) such that, for each $p \in X$, $f(p, t) \in C^2(I)$; then of all periodic partial cubic splines $S_N(p, t)$ with mesh: $0 = t_0 < t_1 < \cdots < t_{i_N} = 1$, $S_N(\{f(p, t_i)\}_i; p, t)$ minimizes

$$\left\| \frac{\partial^2 f(p, t)}{\partial t^2} - \frac{\partial^2 S_N(p, t)}{\partial t^2} \right\|_Y .$$

If $\{\Delta_N\}$ $(N = 1, 2, \cdots)$ is a sequence of meshes on I such that the maximum distance between adjacent points on Δ_N goes to zero as $N \to \infty$, and if $\{S_N(t)\}$ $(N = 1, 2, \cdots)$ is the associated sequence of periodic partial cubic splines coinciding at $\{t_i^N\}$[4] $(i = 0, 1, \cdots, i_N)$ with a function $f(p, t)$ on Y

[4] The superscript N occurring in $\{t_i^N\}$ $(i = 0, 1, \cdots, i_N)$ indicates that t_i^N is a mesh point of Δ_N.

which is in $C^2(I)$ for each $p \in X$; then for each $p \in X$ one-dimensional convergence properties are valid. Normally, convergence is only uniform with respect to t for each p; but if

$$\frac{\partial^2 f(p, t)}{\partial t^2}$$

is uniformly bounded on Y, convergence will be uniform with respect to both p and t simultaneously. In particular, this will be true when X is a compact topological space and

$$\frac{\partial^2 f(p, t)}{\partial t^2}$$

is continuous on Y in the product topology.

For the special case where $X = \{p\}$, $\mu(X) = 1$, $\mu(\phi) = 0$, the theory of partial splines reduces to one-dimensional spline theory. If one forms the integral

$$\int_X S_N(p, t)\, d\mu,$$

it is evident that, as a function of t, the integral is a one-dimensional spline interpolating to

$$\int_X S_N(p, t_i)\, d\mu$$

at $t = t_i$ $(i = 0, 1, \cdots, i_N)$. Similarly if A is a linear operator with domain and range in $L^2\,(X, \mu)$,

$$AS_N(\{f_i(p)\}_i; p, t) = S_N(\{Af_i(p)\}_i; p, t). \tag{1}$$

Here the left hand side of (1) is defined by requiring that $S_N(\{f_i(p)\}_i; p, t)$ be in the domain of A for each $t \in I$. In particular, for $X = [a, b]$, $A_1 = \partial/\partial x$, $A_2 = \partial^2/\partial x^2$ and $f_i(x)$ suitably differentiable with respect to x, we have

$$\frac{\partial}{\partial x} S_N(\{f_i(x)\}_i; x, t) = S_N\left(\left\{\frac{\partial}{\partial x} f_i(x)\right\}_i; x, t\right) \tag{2}$$

and

$$\frac{\partial^2}{\partial x^2} S_N(\{f_i(x)\}_i; x, t) = S_N\left(\left\{\frac{\partial^2}{\partial x^2} f_i(x)\right\}_i; x, t\right). \tag{3}$$

The same type of boundary conditions, however, must be imposed on all splines involved.

The remarks of the preceding paragraph can be made clearer if we consider an explicit representation of a periodic partial cubic spline. In the onedimensional case we have the formulas

$$S_N(\{f_i\}; t) = \frac{M_{i-1}(t_i - t)^3}{6l_i} + \frac{M_i(t - t_{i-1})^3}{6l_i} + \left(\frac{f_i}{l_i} - \frac{M_i l_i}{6}\right)(t - t_{i-1})$$
$$+ \left(\frac{f_{i-1}}{l_i} - \frac{M_{i-1} l_i}{6}\right)(t_i - t)$$
$$(i = 1, 2, \cdots, i_N). \tag{4}$$

where

$$t_{i-1} \leqslant t \leqslant t_i, \qquad l_i = t_i - t_{i-1}, \qquad M_{i_N} = M_0;$$

and

$$\frac{l_i}{6} M_{i-1} + \frac{l_i + l_{i+1}}{3} M_i + \frac{l_{i+1}}{6} M_{i+1} = \frac{f_{i+1} - f_i}{l_{i+1}} - \frac{f_i - f_{i-1}}{l_i},$$
$$(i = 1, 2, \cdots, i_N). \tag{5}$$

These formulas need only be modified to the extent of replacing f_i by $f_i(p)$ in order to obtain a representation for $S_N(\{f_i(p)\}_i; p, t)$. However, the system of equations defined by (5) can be solved and the results can be substituted into (4). This will give the representation

$$S_N(\{f_i(p)\}_i; p, t) = \sum_{k=1}^{i_N} C_{ki}(t) f_k(p) \tag{6}$$

for $t_{i-1} \leqslant t \leqslant t_i$ $(i = 1, 2, \cdots, i_N)$. Since the coefficients $C_{ki}(t)$ are independent of p, the validity of (1), (2), and (3) follows.

Periodic Multidimensional Splines

In considering multidimensional splines, we shall formulate the theory explicitly for cubic splines defined on the unit square which are periodic in each variable. Such splines will be termed doubly-periodic cubic splines. Discussion will be included in the two sections which follow concerning the counterparts of this theory for nonperiodic splines, splines of higher odd degree, and splines of higher dimension. As in the case of one-dimensional splines and partial splines, a function is a spline only with respect to a mesh although it may be a spline with respect to more than one mesh. We shall let M_{QN} denote a mesh on the unit square defined by the set of points $\{(s_i, t_j)\}$ $(i = 0, 1, \cdots, i_Q;\ j = 0, 1 \cdots, j_N)$. For $S_{QN}(s, t)$ to be a spline with respect to the mesh M_{QN}:

(a) in each elementary rectangle $\{s_{i-1} \leqslant s < s_i;\ t_{j-1} \leqslant t \leqslant t_j\}$ $(i = 1, 2, \cdots, i_Q; j = 1, 2, \cdots, j_N)$, $S_{QN}(s, t)$ must be a bicubic;

(b) $\partial^4 S_{QN}(s, t)/\partial s^2 \partial t^2$ must be continuous on the unit square. Here and below we assume differentiation to be independent of order. Consequently, $S_{QN}(s, t)$ is a piecewise bicubic. If $S_{QN}(s, t)$ interpolates to a set of values $\{f_i\}$ $(i = 0, 1, \cdots, i_Q; j = 0, 1, \cdots, j_N)$ at the mesh points of M_{QN} and if we wish to make this dependence clear, we shall use the notation $S_{QN}(\{f_{ij}\}_{ij}; s, t)$ instead of $S_{QN}(s, t)$.

In order to construct $S_{QN}(\{f_{ij}\}_{ij}; s, t)$ for the mesh M_{QN},[5] we first obtain the j_{N+1} one-dimensional periodic cubic splines $S_Q(\{f_{ij}\}_i; s)$ (note that the first and last of these splines are identical). We next take $X = \{s \mid 0 \leqslant s \leqslant 1\}$ and form the periodic partial cubic spline $S_N(\{f_j(s)\}_j; s, t)$ where

$$f_j(s) = S_Q(\{f_{ij}\}_i; s) \qquad (j = 0, 1, \cdots, j_N).$$

We now assert $S_N(\{f_j(s)\}_j; s, t)$ is a spline with respect to the mesh M_{QN}. This can be seen from (6) and from the fact that we have similar representations for each of the j_{N+1} one-dimensional splines $S_Q(\{f_{ij}\}_i; s)$. More specifically, substituting these representations into (6) results in a representation

$$\sum_{i=1}^{i_Q} \sum_{j=1}^{j_N} C_{ij}(s, t) f_{ij}$$

for $S_N(\{f_j(s)\}_j; s, t)$ in each elementary rectangle $\{s_{i-1} \leqslant s \leqslant s_i; t_{j-1} \leqslant t \leqslant t_j\}$ $(i = 1, 2, \cdots, i_Q; j = 1, 2, \cdots, j_N)$. The coefficients C_{ij} are bicubics;[6] consequently (a) is satisfied. The condition (b) can be verified with the aid of (4).

We now show that the periodic spline of interpolation $S_{QN}(\{f_{ij}\}_{ij}; s, t)$ is unique with respect to the mesh M_{QN}, uniqueness being obtained as a consequence of the minimum curvature property which we establish first.

THEOREM 1. *Of all doubly-periodic functions $g(s, t)$ in the unit square which interpolate to the set of values $\{f_{ij}\}$ $(i = 0, 1, \cdots, i_Q; j = 0, 1, \cdots, j_N)$ at the mesh points of a mesh M_{QN} and such that $\partial^4 g(s, t)/\partial s^2 \partial t^2$ is continuous, $S_{QN}(\{f_{ij}\}_{ij}; s, t)$ minimizes*

$$\int_0^1 \int_0^1 \left| \frac{\partial^4 g(s, t)}{\partial s^2 \partial t^2} \right|^2 ds dt. \tag{7}$$

PROOF:

$$\left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} \right\}^2 - \left\{ \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\}^2 = \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} - \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\}^2 + 2 \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} - \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\} \frac{\partial^4 S_{QN}}{\partial^2 \partial t^2}.$$

[5] This construction has been used by us since 1959 for determining cutter centers for milling surfaces in numerically controlled machining.

[6] See the Appendix for a specific representation of these coefficients.

Consequently, if we can show that

$$\int_0^1\int_0^1 \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} - \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\} \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2}\, dsdt \tag{8}$$

is equal to zero, then

$$\int_0^1\int_0^1 \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} \right\}^2 dsdt - \int_0^1\int_0^1 \left\{ \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\}^2 dsdt$$

$$= \int_0^1\int_0^1 \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} - \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\}^2 dsdt \tag{9}$$

and so

$$\int_0^1\int_0^1 \left\{ \frac{\partial^4 g}{\partial s^2 \partial t^2} \right\}^2 dsdt \geqslant \int_0^1\int_0^1 \left\{ \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \right\}^2 dsdt,$$

which will prove the theorem. Integrating (8) twice with respect to t, the first time by parts, yields

$$- \int_0^1 \sum_{j=1}^{j_N} \left(\frac{\partial^2 g}{\partial s^2} - \frac{\partial^2 S_{QN}}{\partial s^2} \right) \frac{\partial^5 S_{QN}}{\partial s^2 \partial t^3} \Bigg|_{t_{j-1}}^{t_j} ds.$$

Repeating the process, this time performing the integrations with respect to s, we obtain for each summand of the form

$$\int_0^1 \left\{ \frac{\partial^2 g(s, t_j)}{\partial s^2} - \frac{\partial^2 S_{QN}(s, t_j)}{\partial s^2} \right\} \frac{\partial^5 S_{QN}(\{f_{ij}\}_{ij}; s, t_j)}{\partial s^2 \partial t^3}\, ds$$

a sum

$$- \sum_{i=1}^{i_Q} \left\{ (g(s, t_j) - S_{QN}(\{f_{ij}\}_{ij}; s, t_j)) \frac{\partial^6 S_{QN}(\{f_{ij}\}_{ij}; s, t_j)}{\partial s^3 \partial t^3} \right\} \Bigg|_{s_{i-1}}^{s_i} .$$

The proof is concluded by observing that these sums vanish since

$$g(s_i , t_j) = S_{QN}(\{f_{ij}\}_{ij}; s_i , t_j) = f_{ij} \qquad (i = 0, 1, \cdots, i_Q; j = 0, 1, \cdots, j_N)$$

The uniqueness of $S_{QN}(\{f_{ij}\}_{ij}; s, t)$ now follows since the difference between any two such doubly-periodic cubic splines is a doubly-periodic cubic spline interpolating to zero at the mesh points of M_{QN}. Thus, in order to give at least as small a value of (7) as the zero function does, it must itself be identically zero. This establishes the following theorem.

THEOREM 2. *$S_{QN}(\{f_{ij}\}_{ij}; s, t)$ is the unique doubly-periodic spline on the mesh M_{QN} which interpolates at the mesh points of M_{QN} to the set of values $\{f_{ij}\}$ $(i = 0, 1, \cdots, i_Q; j = 0, 1, \cdots, j_N)$. In addition,*

$$S_{QN}(\{f_{ij}\}_{ij}; s, t) = S_N(\{S_Q(\{f_{ij}\}_i; s, t) = S_Q(\{S_N(\{f_{ij}\}_j; t)\}_i; s, t).$$

The *best approximation* property for doubly-periodic cubic splines is formulated in the next theorem.

THEOREM 3. *Let $f(s, t)$ be doubly-periodic and $\partial^4 f(s, t)/\partial s^2 \partial t^2$ continuous on the unit square. Then $S_{QN}(\{f(s_i, t_j)\}_{ij}; s, t)$ minimizes*

$$\int_0^1 \int_0^1 \left| \frac{\partial^4 f}{\partial s^2 \partial t^2} - \frac{\partial^4 \hat{S}_{QN}}{\partial s^2 \partial t^2} \right|^2 ds dt \tag{10}$$

over all doubly-periodic cubic splines $\hat{S}_{QN}$ on the mesh M_{QN}. This extremal function is unique to within an additive constant.

PROOF: If in (9) we replace $g(s, t)$ by $f(s, t) - \hat{S}_{QN}(s, t)$ and $S_{QN}(s, t)$ by $\hat{S}_{QN}(s, t) - S_{QN}(\{f(s_i t_j)\}_{ij}; s, t)$, it follows that (10) is minimized.

It should be observed that the best approximation property can serve equally well as a basis for a uniqueness proof. Since the minimal and best approximation properties can not always both be established, this alternative procedure is important. Establishing the uniqueness of Type II splines of interpolation is a case in point.

In order to examine the convergence properties of two-dimensional splines, we introduce the following definitions and terminology. Let $\{M_{QN}\}$ $(Q = 1, 2, \cdots; N = 1, 2, \cdots)$ be a sequence of meshes for the unit square such that M_{QN} is embedded in $M_{Q^1 N^1}$ if $Q^1 \geqslant Q$ and $N^1 \geqslant N$. Let

$$\delta_Q = \max_i | s_i^Q - s_{i-1}^Q |; \qquad \epsilon_N = \max_j (t_j^N - t_{j-1}^N);$$

$$\lambda_Q = \max_i \left| \frac{s_i^Q - s_{i-1}^Q}{s_{i+1}^Q - s_{i-1}^Q} - \tfrac{1}{2} \right|; \qquad \mu_N = \max_j \left| \frac{t_j^N - t_{j-1}^N}{t_{j+1}^N - t_{j-1}^N} - \tfrac{1}{2} \right|.$$

$\{M_{QN}\}$ will be termed *convergent* if $\lim_{Q\to\infty} \delta_Q = 0$ and $\lim_{N\to\infty} \epsilon_N = 0$; *asymptotically uniform in s* if $\lim \lambda_Q = 0$; *asymptotically uniform in t* if $\lim_{N\to\infty} \mu_N = 0$; *and asymptotically uniform* if

$$\lim_{Q\to\infty} \lambda_Q = 0 \qquad \text{and} \qquad \lim_{N\to\infty} \mu_N = 0.$$

We state the following theorem, which is an immediate consequence of one-dimensional spline theory.

THEOREM 4. *Let $f(s, t)$ be a doubly-periodic function on the unit square such that $\partial^4 f(s, t)/\partial s^2 \partial t^2$ is continuous. Let $\{M_{QN}\}$ $(Q = 1, 2, \cdots)$ be a sequence of meshes with $\delta_Q \to 0$ as $Q \to \infty$. The following relations are true and the limits exist uniformly with respect to (s, t):*

1. $\lim_{Q\to\infty} S_{QN}(\{f(s_i^Q, t_j^N)\}_{ij}; s, t) = S_N(\{f(s, t_j^N)\}_j; s, t)$;

2. $\lim_{Q\to\infty} \frac{\partial}{\partial s} S_{QN}(\{f(s_i^Q, t_j^N)\}_{ij}; s, t) = S_N\left(\left\{\frac{\partial}{\partial s} f(s, t_j^N)\right\}_j; s, t\right)$;

3. $\lim_{Q\to\infty} \left\| \frac{\partial^2}{\partial s^2} S_{QN}(\{f(s_i^Q, t_j^N)\}_{ij}; s, t) - S_N\left(\left\{\frac{\partial^2 f}{\partial s^2}(s, t_j^N)\right\}_j; s, t\right) \right\|_Y = 0$;

4. *if $\lambda_Q \to 0$ as $Q \to \infty$*

$$\lim_{Q\to\infty} \frac{\partial^2}{\partial s^2} S_{QN}(\{f(s_i^Q, t_j^N)\}_{ij}; s, t) = S_N\left(\left\{\frac{\partial^2 f(s_i t_j^N)}{\partial s^2}\right\}_j; s, t\right).$$

COROLLARY. *The roles of s and t are interchangeable in Theorem 4.*

With respect to taking iterated limits we have the following theorem.

THEOREM 5. *Let $f(s, t)$ be a doubly-periodic function on the unit square such that $\partial^4 f(s, t)/\partial s^2 \partial t^2$ is continuous. Let $\{M_{QN}\}$ $(Q = 1, 2, \cdots; N = 1, 2, \cdots)$ be a convergent sequence of meshes. The following relations are true and the limits exist uniformly with respect to $(s, t) \in Y$. $S_{QN}(s, t)$ denotes $S_{QN}(\{f(s_i^Q, t_j^N)\}_{ij}; s, t)$ in these relations.*

1. $$\lim_{Q\to\infty} \lim_{N\to\infty} S_{QN}(s, t) = \lim_{N\to\infty} \lim_{Q\to\infty} S_{QN}(s, t) = f(s, t),$$

2. $$\lim_{Q\to\infty} \lim_{N\to\infty} \frac{\partial}{\partial s} S_{QN}(s, t) = \lim_{N\to\infty} \lim_{Q\to\infty} \frac{\partial}{\partial s} S_{QN}(s, t) = \frac{\partial}{\partial s} f(s, t),$$

3. $$\lim_{Q\to\infty} \lim_{N\to\infty} \frac{\partial^2}{\partial s \partial t} S_{QN}(s, t) = \lim_{N\to\infty} \lim_{Q\to\infty} \frac{\partial^2}{\partial s \partial t} S_{QN}(s, t) = \frac{\partial^2}{\partial s \partial t} f(s, t),$$

4. $$\lim_{Q\to\infty} \lim_{N\to\infty} \left\| \frac{\partial^3}{\partial s^2 \partial t} S_{QN}(s, t) - \frac{\partial^3}{\partial s^2 \partial t} f(s, t) \right\|_Y$$
$$= \lim_{N\to\infty} \lim_{Q\to\infty} \left\| \frac{\partial^3}{\partial s^2 \partial t} S_{QN}(s, t) - \frac{\partial^3}{\partial s^2 \partial t} f(s, t) \right\|_Y = 0,$$

5. $$\lim_{Q\to\infty} \lim_{N\to\infty} \left\| \frac{\partial^4}{\partial s^2 \partial t^2} S_{QN}(s, t) - \frac{\partial^4}{\partial s^2 \partial t^2} f(s, t) \right\|_Y$$
$$= \lim_{N\to\infty} \lim_{Q\to\infty} \left\| \frac{\partial^4}{\partial s^2 \partial t^2} S_{QN}(s, t) - \frac{\partial^4}{\partial s^2 \partial t^2} f(s, t) \right\|_Y = 0,$$

6. *if* $\lambda_Q \to 0$ *as* $Q \to \infty$,

$$\lim_{Q\to\infty}\lim_{N\to\infty}\frac{\partial^3}{\partial s^2\partial t}S_{QN}(s,t)=\lim_{N\to\infty}\lim_{Q\to\infty}\frac{\partial^3}{\partial s^2\partial t}S_{QN}(s,t)=\frac{\partial^3}{\partial s^2\partial t}f(s,t),$$

7. *if* $\lambda_Q \to 0$ *as* $Q \to \infty$ *and* $\mu_N \to 0$ *as* $N \to \infty$,

$$\lim_{Q\to\infty}\lim_{N\to\infty}\frac{\partial^4}{\partial s^2\partial t^2}S_{QN}(s,t)=\lim_{N\to\infty}\lim_{Q\to\infty}\frac{\partial^4}{\partial s^2\partial t^2}S_{QN}(s,t)=\frac{\partial^4}{\partial s^2\partial t^2}f(s,t).$$

PROOF: We shall consider the proof of relation 7 only as it is illustrative of the most general form of the argument. Let

$$L_{QN}=\frac{\partial^4}{\partial s^2\partial t^2}S_{QN}(\{f(s_i^Q,t_j^N)\}_{ij};s,t).$$

By Theorem 2, we have

$$L_{QN}=\frac{\partial^4}{\partial s^2\partial t^2}S_N(\{S_Q(\{f(s_i^Q,t_j^N)\}_i;s)\}_j;s,t),$$

and, in view of (3),

$$L_{QN}=\frac{\partial^2}{\partial t^2}S_N\left(\left\{\frac{\partial^2}{\partial s^2}S_Q(\{f(s_i^Q,t_j^N)\}_i;s)\right\}_j;s,t\right).$$

Since as $Q \to \infty$ the dependence on t is invariant,

$$\lim_{Q\to\infty}L_{QN}=\frac{\partial^2}{\partial t^2}S_N\left(\left\{\lim_{Q\to\infty}\frac{\partial^2}{\partial s^2}S_Q(\{f(s_i^Q,t_j^N)\}_i;s)\right\}_j;s,t\right).$$

By one-dimensional theory,

$$\lim_{Q\to\infty}\frac{\partial^2}{\partial s^2}S_Q(\{f(s_i^Q,t_j^N\}_i;s)=\frac{\partial^2}{\partial s^2}f(s,t_j^N).$$

Consequently,

$$\lim_{N\to\infty}\lim_{Q\to\infty}L_{QN}=\lim_{N\to\infty}\frac{\partial^2}{\partial t^2}S_N\left(\left\{\frac{\partial^2}{\partial s^2}f(s,t_j^N)\right\}_j;s,t\right)=\frac{\partial^4}{\partial t^2\partial s^2}f(s,t),$$

due to the convergence properties of partial splines. Reversing the roles of s and t leads to the result that

$$\lim_{Q\to\infty}\lim_{N\to\infty}\frac{\partial^4}{\partial s^2\partial t^2}S_{QN}(\{f(s_i^Q,t_j^N)\}_{ij};s,t)=\frac{\partial^4}{\partial s^2\partial t^2}f(s,t),$$

from which relation 7 follows. In addition, $|\partial^4 f(s,t)/\partial t^2 \partial s^2|$ is uniformly bounded with respect to $(s,t) \in Y$ which implies that convergence is uniform with respect to $(s,t) \in Y$. Due to this uniformity, the iterated limits are, of course, both equal to the double limit although we have not expressed this fact in the formal statement of the theorem.

COROLLARY. *The roles of s and t are interchangeable in Theorem 5.*

NONPERIODIC CUBIC SPLINES

As in the case of one-dimensional splines, it is advantageous to classify functions on the unit square according to boundary conditions imposed on them on a subset (proper or improper) of the boundary of the square. The conditions of greatest interest for spline theory are

$$\mathrm{I}\begin{cases} \dfrac{\partial S_{QN}}{\partial t} \text{ is prescribed at } s = s_i \ (i = 0, 1, \cdots, i_Q);\ t = 0,\ t = 1; \\[2ex] \dfrac{\partial S_{QN}}{\partial s} \text{ is prescribed at } t = t_j \ (j = 0, 1, \cdots, j_N);\ s = 0,\ s = 1; \\[2ex] \dfrac{\partial^2 S_{QN}}{\partial s \partial t} \text{ is prescribed at } (0,0), (0,1), (1,0), \text{ and } (1,1); \end{cases}$$

and

$$\mathrm{II}\begin{cases} \dfrac{\partial^2 S_{QN}}{\partial t^2} \text{ is prescribed at } s = s_i \ (i = 0, 1, \cdots, i_Q);\ t = 0,\ t = 1; \\[2ex] \dfrac{\partial^2 S_{QN}}{\partial s^2} \text{ is prescribed at } t = t_j \ (j = 0, 1, \cdots, j_N);\ s = 0,\ s = 1; \\[2ex] \dfrac{\partial^4 S_{QN}}{\partial s^2 \partial t^2} \text{ is prescribed at } (0,0), (0,1), (1,0), \text{ and } (1,1). \end{cases}$$

We shall refer to functions satisfying I or II as Type I functions or Type II functions respectively. When all prescribed values are zero, we shall employ the prime notation just as in the one-dimensional case. Conditions I are those considered by deBoor [11].

Obviously, this classification does not separate functions into disjoint classes, but in the case of spline functions it does lead to explicit representations and allows the establishment of existence and uniqueness theorems. Again, it is useful to separate Type I and Type II functions into equivalence classes modulo Type I′ and Type II′ functions, respectively.

By means of the following general identity, the establishment of minimum curvature and best approximation properties is obtained for non-periodic two-dimensional cubic splines in the same manner as in the periodic case.

$$\begin{aligned}
\int_0^1\int_0^1 \left\{\frac{\partial^4 g}{\partial s^2 \partial t^2}\right\}^2 dsdt - \int_0^1\int_0^1 \left\{\frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2}\right\}^2 dsdt &= \int_0^1\int_0^1 \left\{\frac{\partial^4 g}{\partial s^2 \partial t^2} - \frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2}\right\}^2 dsdt \\
&\quad - 2\sum_{i=1}^{i_Q} \left\{\left(\frac{\partial g}{\partial t} - \frac{\partial S_{QN}}{\partial t}\right)\frac{\partial^5 S_{QN}}{\partial s^3 \partial t^2}\right\}\Bigg|_{s_{i-1}}^{s_i}\Bigg|_{t=0}^{t=1} \\
&\quad - 2\sum_{j=1}^{j_N} \left\{\left(\frac{\partial g}{\partial s} - \frac{\partial S_{QN}}{\partial s}\right)\frac{\partial^5 S_{QN}}{\partial s^2 \partial t^3}\right\}\Bigg|_{t_{j-1}}^{t_j}\Bigg|_{s=0}^{s=1} \\
&\quad + 2\left\{\left(\frac{\partial^2 g}{\partial s \partial t} - \frac{\partial^2 S_{QN}}{\partial s \partial t}\right)\frac{\partial^4 S_{QN}}{\partial s^2 \partial t^2}\right\}\Bigg|_{t=0}^{t=1}\Bigg|_{s=0}^{s=1} \\
&\quad + 2\sum_{j=1}^{j_N}\sum_{i=1}^{i_Q} \left\{(g - S_{QN})\frac{\partial^6 S_{QN}}{\partial s^3 \partial t^3}\right\}\Bigg|_{s_{i-1}}^{s_i}\Bigg|_{t_{j-1}}^{t_j}. \qquad (11)
\end{aligned}$$

Thus, for each Type I equivalence class we have minimum curvature and best approximation properties and for each Type II equivalence class we have a best approximation property. Type II′ splines have the strongest minimum curvature properties and for a given function $g(s, t)$ the strongest best approximation properties are provided by splines in the same Type I equivalence class as $g(s, t)$.

Generalizations

There is a direct extension to two-dimensional splines of higher odd degree in each variable in terms of the measure

$$\int_0^1\int_0^1 \left|\frac{\partial^{m+k+2} g(s, t)}{(\partial s)^{m+1}\,(\partial t)^{k+1}}\right|^2 dsdt;$$

but the analog of (11) is complicated. The extension is useful if $S_{QN}(s, t)$ exists and is unique and where we have convergence of the one-dimensional splines and partial splines entering into the proof. Thus, most cases of interest are covered. In particular this is true when $S_{QN}(s, t)$ is a two-dimensional spline of degree $2k + 1$ in t and $2m + 1$ in s with either I, II, or III.

I
$\dfrac{\partial^\alpha S_{QN}}{\partial t^\alpha}$ is prescribed for $\alpha = 1, 2, \cdots, k$ at $s = s_i$ $(i = 0, 1, \cdots, i_Q)$, $t = 0$, $t = 1$;

$\dfrac{\partial^\alpha S_{QN}}{\partial s^\alpha}$ is prescribed for $\alpha = 1, 2, \cdots, m$ at $t = t_j$ $(j = 0, 1, \cdots, j_N)$, $s = 0$, $s = 1$; and

$\dfrac{\partial^\alpha S_{QN}}{\partial s^\beta \partial t^\gamma}$ is prescribed for $\alpha = \beta + \gamma$, $\beta = 1, 2, \cdots, m$, $\gamma = 1, 2, \cdots, k$ at $(s, t) = (0, 0), (0, 1), (1, 0), (1, 1)$.

II
$\dfrac{\partial^\alpha S_{QN}}{\partial t^\alpha}$ is prescribed for $\alpha = k + 1, k + 2, \cdots, 2k$ at $s = s_i$ $(i = 0, 1, \cdots, i_Q)$, $t = 0$, $t = 1$;

$\dfrac{\partial^\alpha S_{QN}}{\partial s^\alpha}$ is prescribed for $\alpha = m + 1, m + 2, \cdots, 2m$ at $t = t_j$ $(j = 0, 1, \cdots, j_N)$, $s = 0$, $s = 1$; and

$\dfrac{\partial^\alpha S_{QN}}{\partial s^\beta \partial t^\gamma}$ is prescribed for $\alpha = \beta + \gamma$, $\beta = m + 1, m + 2, \cdots, 2m$, $\gamma = k + 1, k + 2, \cdots, 2k$ at $(s, t) = (0, 0), (0, 1), (1, 0)$, and $(1, 1)$.

III S_{QN} is a doubly periodic spline of degree $2k + 1$ in t and $2m + 1$ in s.

These boundary conditions lead to a natural classification of functions, and the arguments needed to establish minimum curvature, best approximation, and convergence properties are essentially the same as in the situations already examined.

One also can construct inductively a theory of higher dimensional splines. The use of partial splines to add an additional dimension proceeds just as in the transition from one-dimensional splines to two-dimensional splines. For an n-dimensional spline of degree $2k + 1$ in each variable the measure employed is

$$\int_V \left| \frac{\partial^{n(k+1)}}{(\partial x_1)^{k+1} \cdots (\partial x_n)^{k+1}} f(x_1, \cdots, x_n) \right|^2 dx_1\, dx_2 \cdots dx_n ,$$

where V is the n-dimensional cube.

Orthogonality

In the remainder of this paper we develop the orthogonality properties of two-dimensional cubic splines. The development will be explicit only for the periodic case, but the extension to Type I and Type II splines is immediate. The extension to splines of higher degree offers no difficulty; the extension to higher dimensions offers mainly notational difficulties. The development of the orthogonality properties of one-dimensional splines is carried through in [12].

The doubly-periodic functions $f(s, t)$ on the unit square for which $(\partial^4/\partial s^2\partial t^2)f(s, t)$ exists and is continuous form a Hilbert space H under the pseudonorm

$$\|f\|_H = \left\{\int_0^1\int_0^1 \left| \frac{\partial^4 f(s, t)}{\partial s^2 \partial t^2} \right|^2 dsdt\right\}^{1/2}.$$

We shall now show that H has a decomposition

$$H = \sum_{i=1}^{\infty} \oplus H_i \tag{12}$$

into a direct sum of finite dimensional vector spaces H_i which are mutually orthogonal and where each H_i consists entirely of spline functions. In general, a mesh will be associated with each H_i and the elements of H_i will all be splines with respect to this mesh. The decomposition (12) can be refined, in standard fashion, to introduce into H an orthonormal basis consisting entirely of spline functions. Convergence with respect to $\|\cdot\|_H$, however, will only be unique up to a constant function due to the pseudo nature of the norm. The representation of an element $f \in H$ in terms of this basis is unique if the representation is required to interpolate to f at the mesh points of the meshes associated with the decomposition. Under these conditions the representation of f in terms of the basis will converge uniformly to f, and $\partial f/\partial s$, $\partial f/\partial t$, and $\partial^2 f/\partial s\partial t$ will be obtainable by termwise differentiation. When the associate meshes are asymptotically uniform, in one or both variables as the case demands, $\partial^4 f/\partial s^2\partial t^2$, $\partial^3 f/\partial s\partial t^2$, and $\partial^3 f/\partial s^2\partial t$ are also obtainable by termwise differention.

Let M_{QN} be embedded in $M_{Q^1N^1}$ and let $S_{QN}(s, t)$ and $S_{Q^1N^1}(s, t)$ be doubly periodic two-dimensional cubic splines defined on M_{QN} and $M_{Q^1N^1}$ respectively. Let $S_{Q^1N^1}$ have the additional property that it vanishes at the mesh points of M_{QN}. Consider the integral

$$\int_0^1\int_0^1 \left\{\frac{\partial^4 S_{Q^1N^1}}{\partial s^2\partial t^2} \cdot \frac{\partial^4 S_{QN}}{\partial s^2\partial t^2}\right\} dsdt. \tag{13}$$

Express (13) as a sum of integrals each taken over an elementary rectangle of M_{QN}. Integrating twice with respect to s, the first time by parts, one obtains

$$-\sum_{j=1}^{j_N}\sum_{i=1}^{i_Q}\int_{t_{j-1}^N}^{t_j^N} \left(\frac{\partial^2 S_{Q^1N^1}}{\partial t^2} \cdot \frac{\partial^5 S_{QN}}{\partial s^3\partial t^2}\right)\Bigg|_{s_{i-1}^Q}^{s_i^Q} ds.$$

Repeating the process, this time with respect to t, leads to a sum of the general form

$$\sum_{i,j} C_{ij} S_{Q^1 N^1}(s_i{}^Q, t_j{}^N),$$

which vanishes since $S_{Q^1N^1}$ vanishes at the mesh points of M_{QN}. We have thus shown that S_{QN} and $S_{Q^1N^1}$ are orthogonal with respect to the inner product

$$(f, g) = \int_0^1 \int_0^1 \left\{ \frac{\partial^4 f}{\partial s^2 \partial t^2} \cdot \frac{\partial^4 g}{\partial s^2 \partial t^2} \right\} ds dt.$$

As a consequence, if $\{M_{QN}\}$ $(Q = 1, 2, \cdots; N = 1, 2, \cdots)$ is a convergent sequence such that $M_{Q^1N^1}$ is embedded in M_{QN} for $Q^1 \leqslant Q$, $N^1 \leqslant N$, then

$$H = \sum_{Q=1}^{\infty} \sum_{N=1}^{\infty} \oplus V_{QN}$$

where V_{QN} is the family of all doubly-periodic two-dimensional splines on M_{QN} which vanish at the mesh points of any mesh in $\{M_{QN}\}$ embedded in M_{QN}.

Letting $H_1 = V_{11}$, $H_2 = V_{21}$, $H_3 = V_{12}$, etc., (12) is established. Since

$$\sum_{Q=1}^{Q_0} \sum_{N=1}^{N_0} \oplus V_{QN}$$

is precisely the family of all doubly-periodic two-dimensional splines on the mesh $M_{Q_0N_0}$, the pointwise convergence and termwise differention properties of the series representation of $f \in H$ in terms of an orthonormal basis based on this decomposition of H follow from Theorem 4.

Some interesting orthonormal bases, consistent with this decomposition of H, are developed for one-dimensional splines in [12]. This is done by replacing a given sequence of meshes $\{M_{QN}\}$ $(Q = 1, 2, \cdots; N = 1, 2, \cdots)$ by a sequence which introduces only one new mesh point at a time. The analogue for two dimensions is complicated by the fact that the addition of a single new mesh point unbalances the rectangular character of the meshes. This difficulty can be eliminated by approaching the problem from the point of view that H is the direct product of two vector spaces each possessing a basis consisting of one-dimensional splines in the variables s and t, respectively. Two-dimensional splines and their higher dimensional analogues provide a means of obtaining orthonormal families of functions for very general regions, a property which has not yet been explored.

Since every $f \in H$ has a representation,

$$f = \sum_{Q=1}^{\infty} \sum_{N=1}^{\infty} f_{QN},$$

where $f_{QN} \in V_{QN}$, it follows that

$$\int_0^1 \int_0^1 \left| \frac{\partial^4 f}{\partial s^2 \partial t^2} \right|^2 ds dt = \sum_{Q=1}^{\infty} \sum_{N=1}^{\infty} \int_0^1 \int_0^1 \left| \frac{\partial^4 f_{QN}}{\partial s^2 \partial t^2} \right|^2 ds dt.$$

This Parseval-type relation, which emphasizes the monotonic nature of both the minimum curvature and best approximation properties, persists for most varieties of splines, particularly those of greatest significance.

APPENDIX

Periodic Two-Dimensional Splines

The unit square $0 \leqslant s \leqslant 1, 0 \leqslant t \leqslant 1$ is divided into elemental rectangles by subdivisions $0 < s_1 < s_2 < \cdots < s_p = 1$, $0 < t_1 < t_2 < \cdots < t_q = 1$. Set

$$h_i = s_i - s_{i-1}, \qquad k_i = t_i - t_{i-1},$$

$$\lambda_i = \frac{h_{i+1}}{h_i + h_{i+1}}, \qquad \mu_i = \frac{k_{i+1}}{k_i + k_{i+1}}.$$

We wish to construct the periodic bicubic $S_{pq}(s, t)$ over the unit square such that for a prescribed set of values $f_{i,j}$ $(i = 1, 2, \cdots, p;\ j = 1, 2, \cdots, q)$, $S_{p,q}(x_i, y_j) = f_{i,j}$.

First construct one-dimensional splines at each level $t = t_j$ $(j = 1, 2, \cdots, q)$. Call the resulting spline $S_j(s)$: for $s_{i-1} \leqslant s \leqslant s_i$,

$$S_j(s) = M_{i-1,j} \frac{(s_i - s)^3}{6h_i} + M_{i,j} \frac{(s - s_{i-1})^3}{6h_i} + \left(f_{i-1,j} - \frac{h_i^2}{6} M_{i-1,j} \right) \frac{s_i - s}{h_i}$$

$$+ \left(f_{i,j} - \frac{h_i^2}{6} M_{i,j} \right) \frac{s - s_{i-1}}{h_i},$$

where

$$A \cdot \begin{pmatrix} M_{1j} \\ M_{2j} \\ \vdots \\ M_{pj} \end{pmatrix} = \Delta_h \cdot \begin{pmatrix} f_{1j} \\ f_{2j} \\ \vdots \\ f_{pj} \end{pmatrix},$$

$$A = \begin{pmatrix} \frac{2}{3} & \frac{\lambda_1}{3} & 0 & \cdots & 0 & \frac{1-\lambda_1}{3} \\ \frac{1-\lambda_2}{3} & \frac{2}{3} & \frac{\lambda_2}{3} & \cdots & 0 & 0 \\ & \vdots & & & \vdots & \\ \frac{\lambda_p}{3} & 0 & 0 & \cdots & \frac{1-\lambda_p}{3} & \frac{2}{3} \end{pmatrix},$$

$$\Delta_h = \begin{pmatrix} -\frac{2}{h_1 h_2} & \frac{2}{h_2(h_1+h_2)} & 0 \quad 0 & & \\ & & \cdots & 0 & \frac{2}{h_1(h_1+h_2)} \\ \frac{2}{h_2(h_2+h_3)} & \frac{-2}{h_2 h_3} & \frac{2}{h_3(h_2+h_3)} & & \\ & & \cdots & 0 & 0 \\ \vdots & & & \vdots & \\ \frac{2}{h_1(h_p+h_1)} & 0 & 0 & & \\ & & \cdots & \frac{2}{h_p(h_p+h_1)} & \frac{-2}{h_p h_1} \end{pmatrix}.$$

Designate by M the matrix (M_{ij}) formed in this way and set $f = (f_{ij})$. The matrix M may be obtained by a simple algorithm without formally inverting the matrix A (see [3], p. 226). We have

$$M = A^{-1}\Delta_h f. \tag{1}$$

At each location we have a resulting set of values, $S_1(s)$, $S_2(s)$, $\cdots$, $S_p(s)$. Construct the periodic spline interpolating to these values at the location t_j, obtaining, for $t_{j-1} \leqslant t \leqslant t_j$,

$$S_{pq}(s, t) = N_{j-1}(s)\frac{(t_j - t)^3}{6k_j} + N_j(s)\frac{(t - t_{j-1})^3}{6k_j} + \left(S_{j-1}(s) - \frac{k_j^2}{6}N_{j-1}(s)\right)$$
$$\cdot \frac{t_j - t}{k_j} + \left(S_j(s) - \frac{k_j^2}{6}N_j(s)\right)\frac{t - t_{j-1}}{k_j}.$$

Here $N_j(s)$ represents $\partial^2 S_{pq}(s, t_j)/\partial t^2$. If B is the coefficient matrix on the t-mesh analogous to A on the s-mesh, and if Δ_k is the second-difference matrix in k analogous to Δ_h, then with $\mathbf{M}_i = (M_{i,1}, M_{i,2}, \cdots, M_{iq})$, $\mathbf{f}_i = (f_{i,1}, f_{i,2}, \cdots f_{i,q})$, we have

$$\begin{pmatrix} N_1(s) \\ N_2(s) \\ \vdots \\ N_q(s) \end{pmatrix} = B^{-1}\Delta_k \begin{pmatrix} S_1(s) \\ S_2(s) \\ \vdots \\ S_q(s) \end{pmatrix} = B^{-1}\Delta_k \left[\frac{(s_i - s)^3}{6h_i} \mathbf{M}_{i-1}^T + \frac{(s - s_{i-1})^3}{6h_i} \mathbf{M}_i^T \right.$$
$$\left. + \left(\mathbf{f}_{i-1}^T - \frac{h_i^2}{6} \mathbf{M}_{i-1}^T \right) \frac{s_i - s}{h_i} + \left(\mathbf{f}_i^T - \frac{h_i^2}{6} \mathbf{M}_i^T \right) \frac{s - s_{i-1}}{h_i} \right].$$

Set

$$N = (N_{ij}) = f(B^{-1}\Delta_k)^T, \qquad G = (g_{ij}) = A^{-1}\Delta_h f(B^{-1}\Delta_k)^T. \tag{2}$$

It follows, for $s_{i-1} \leqslant s \leqslant s_i$, $t_{j-1} \leqslant t \leqslant t_j$, that

$$\begin{aligned}
S_{pq}(s, t) = {} & \frac{(t_j - t)^3}{6k_j} \left\{ g_{i-1,j-1} \frac{(s_i - s)^3}{6h_i} + g_{i,j-1} \frac{(s - s_{i-1})^3}{6h_i} \right. \\
& \left. + \left(N_{i-1,j-1} - \frac{h_i^2}{6} g_{i-1,j-1} \right) \frac{s_i - s}{h_i} + \left(N_{i,j-1} - \frac{h_i^2}{6} g_{i,j-1} \right) \frac{s - s_{i-1}}{h_i} \right\} \\
& + \frac{(t - t_{j-1})^3}{6k_j} \left\{ g_{i-1,j} \frac{(s_i - s)^3}{6h_i} + g_{i,j} \frac{(s - s_{i-1})^3}{6h_i} \right. \\
& \left. + \left(N_{i-1,j} - \frac{h_i^2}{6} g_{i-1,j} \right) \frac{s_i - s}{h_i} + \left(N_{i,j} - \frac{h_i^2}{6} g_{i,j} \right) \frac{s - s_{i-1}}{h_i} \right\} \\
& + \frac{t_j - t}{k_j} \left\{ \left(M_{i-1,j-1} - \frac{k_j^2}{6} g_{i-1,j-1} \right) \frac{(s_i - s)^3}{6h_i} \right. \\
& + \left(M_{i,j-1} - \frac{k_j^2}{6} g_{i,j-1} \right) \frac{(s - s_{i-1})^3}{6h_j} \\
& + \left[f_{i-1,j-1} - \frac{h_i^2}{6} M_{i-1,j-1} - \frac{k_j^2}{6} \left(N_{i-1,j-1} - \frac{h_i^2}{6} g_{i-1,j-1} \right) \right] \frac{s_i - s}{h_i} \\
& \left. + \left[f_{i,j-1} - \frac{h_i^2}{6} M_{i,j-1} - \frac{k_j^2}{6} \left(N_{i,j-1} - \frac{h_i^2}{6} g_{i,j-1} \right) \right] \frac{s - s_{i-1}}{h_i} \right\} \\
& + \frac{t - t_{j-1}}{k_j} \left\{ \left(M_{i-1,j} - \frac{k_j^2}{6} g_{i-1,j} \right) \frac{(s_i - s)^3}{6h_i} \right. \\
& + \left(M_{i,j} - \frac{k_j^2}{6} g_{i,j} \right) \frac{(s - s_{i-1})^3}{6h_i} \\
& + \left[f_{i-1,j} - \frac{h_i^2}{6} M_{i-1,j} - \frac{k_j^2}{6} \left(N_{i-1,j} - \frac{h_i^2}{6} g_{i-1,j} \right) \right] \frac{s_i - s}{h_i} \\
& \left. + \left[\left(f_{i,j} - \frac{h_i^2}{6} M_{i,j} \right) - \frac{k_j^2}{6} \left(N_{i,j} - \frac{h_i^2}{6} g_{i,j} \right) \right] \frac{s - s_{i-1}}{h_i} \right\}.
\end{aligned} \tag{3}$$

It is readily verified that

$$\frac{\partial^2 S_{pq}(s_i\,,\,t_j)}{\partial s^2} = M_{i,j}\,, \qquad \frac{\partial^2 S_{pq}(s_i\,,\,t_j)}{\partial t^2} = N_{i,j}$$

as was to be expected, and that

$$\frac{\partial^4 S_{pq}(s_i\,,\,t_j)}{\partial s^2 \partial t^2} = g_{i,j}\,.$$

If we set $\xi_i = (s - s_{i-1})/h_i$, $\eta_j = (t - t_{j-1})/k_j$, then

$$\begin{aligned} S_{pq}(s,t) = (1-\xi_i)(1-\eta_j)\,\xi_i\eta_j \Big\{ & \frac{h_i^2 k_j^2}{36}\,[g_{i-1,j-1}(2-\xi_i)(2-\eta_j) \\ & + g_{i-1,j}(2-\xi_i)(1+\eta_j) + g_{i,j-1}(1+\xi_i)(2-\eta_j) + g_{i,j}(1+\xi_i)(1+\eta_j)] \\ & - \frac{h_i^2}{6}\left[M_{i-1,j-1}\frac{2-\xi_i}{\eta_j} + M_{i-1,j}\frac{2-\xi_i}{1-\eta_j} + M_{i,j-1}\frac{1+\xi_i}{\eta_j} + M_{i,j}\frac{1+\xi_i}{1-\eta_j}\right] \\ & - \frac{k_j^2}{6}\left[N_{i-1,j-1}\frac{2-\eta_j}{\xi_i} + N_{i,j-1}\frac{2-\eta_j}{1-\xi_i} + N_{i-1,j}\frac{1+\eta_j}{\xi_i} + N_{i,j}\frac{1+\eta_j}{1-\xi_i}\right] \\ & + \frac{f_{i-1,j-1}}{\xi_i\eta_j} + \frac{f_{i-1,j}}{\xi_i(1-\eta_j)} + \frac{f_{i,j-1}}{(1-\xi_i)\,\eta_j} + \frac{f_{i,j}}{(1-\xi_i)(1-\eta_j)}\Big\}\,. \end{aligned}$$

We note in passing the symmetry in (1), (2), and (3) which indicates the independence of the order of the fitting.

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